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A game theory formulation of decision making under conditions of uncertainty and risk

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ABSTRACT

A game setting is developed for decision making under conditions of uncertainty and risk for a general class of problems. The methodology is illustrated using assumptions governing future oil prices and environmental degradation to evaluate long term investment alternatives.

1. Introduction/outline

The essence of a decision is the decision maker's personal balance of expected payoff and risk. A game theory formulation of decision making under conditions of uncertainty and risk, made possible by an appeal to the Central Limit Theorem, allows us to extend our basic understanding of decision making to a large class of complex decision problems with time as an independent variable.

In general, the decision maker plays against NATURE in a single move game. At the time the decision maker moves, NATURE's time dependent strategy is hidden forcing the decision maker to consider NATURE's strategy as uncertain. The formulation presented here maintains a distinction between uncertainty modeled without assuming a distribution and risk arising from quantifiable random variability, i.e., with a distribution.

A standard definition of game theory:

"Game theory is the study of the ways in which strategic interactions among rational players produce outcomes with respect to the preferences (or utilities) of those players, none of which might have been intended by any of them." (Stanford Encyclopedia of Philosophy)

provides motivation for our development and indicates the hopes we have for its application.

As an illustrative example, we will demonstrate how assumptions governing future oil prices and environmental degradation can be used to evaluate long term investment alternatives in a game setting, supporting strategic thinking in a real world problem scenario. We are presenting a complete decision methodology. Using the example, we will show how easily and transparently the modeling and decision methodology can be implemented.

Outline of the paper: Section 2. An introduction to the problem of evaluating long term investments in a decision environment of increasing oil prices and environmental degradation. This decision problem both motivates and illustrates our modeling/decision methodology.

Section 3. We are faced with two tasks: making sense of this unexplored complex investment environment, i.e., constructing meaningful models of the decision environment accounting for what we know, what we are willing to assume, and what is unknown, and developing a rational decision methodology based on the models. In Section 3 we develop a family of investment models.

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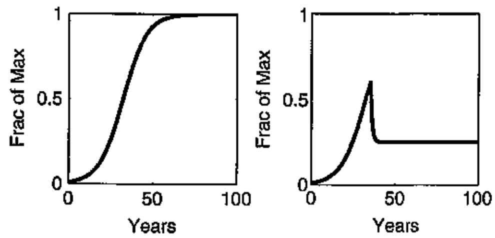

Fig. 1. The two graphs represent possible scenarios for future oil prices (or measures of environmental degradation). In the first scenario social change would be incremental. In the second, at some point change would be abrupt, the system would bifurcate.

Section 4. In this section we develop a family of general decision models. Basically, we introduce a two person game with players NATURE and PLAYER. NATURE's strategies are future oil prices and environmental degradation. PLAYER's strategies are rational investment decisions. We provide a complete computational example illustrating the construction of the decision models.

Section 5. We introduce a decision methodology which is an extension of multi-criteria decision analysis, replacing random criteria values with deterministic surrogates specifying risk. The criteria and surrogates are functions of the uncertainties. The decision goal is a balance of expected payoffs and risk.

Section 6. In the concluding remarks we address questions of more than two uncertainties and more than one criteria for evaluating an investment.

2. Possible scenarios for future oil prices and air quality degradation and the relationship of these scenarios to long term investment decisions

Peak world oil production and "the tipping point" for global environmental degradation will trigger a new era of rapidly increasing oil prices and environmental degradation initiating large social changes on a global scale. The emerging Chinese and Indian industrial economies will accelerate demand for an ever smaller energy resource resulting in higher oil prices. Also, their willingness to accept environmental degradation in trade for economic development will contribute to an accelerated deterioration of the environment with global consequences. Numerous sources point to the critical nature of these emergent realities. (Google "peak oil" or "tipping point".)

Cheap oil and the acceptance of the idea of a boundless natural world have come to an end. We are entering a new era where the old signposts have become unreliable. There is no accepted roadmap for our journey from this point into the future. Should we seek to moderate oil prices by exploiting heavy crude to the detriment of the environment? Or, should we regulate the use of carbon based fuels in an attempt to remediate the environment? Is there a framework for thinking strategically about the rapidly changing investment environment that we must confront? Is there a rational decision methodology for choosing among present investment alternatives knowing that big changes are in the future?

Assuming a continuation of the global market economy has consequences. As the price of oil rises, perhaps to levels that are multiples of current prices, and as environmental degradation begins to have a global impact on human health and mortality, economically viable alternatives to the way we currently live will emerge. The only "give" enabling an adaptation to this new reality and the continuation of the global market economy are large scale changes in the way we live, eat, and work. These changes will create investment opportunities but also introduce uncertainties and new sources of risk. Economic dynamics will force the social changes with new winners and losers.

Nothing in the real world increases at the same or higher rate without limit. Otherwise, we would not be here to speculate about the future. Countervailing forces will always emerge. In the case of oil prices, economic forces will force social changes which will cap the price. In order to simplify the presentation, we will concentrate on the scenarios illustrated by the first graph in Fig. 1.

NATURE's strategies. Long term investing in this new era can be thought of as a game we play against NATURE. NATURE's strategy will be future oil prices and environmental degradation. At the time of our investment decision we will not know which strategy NATURE has chosen. We must evaluate our investment alternatives in terms of assumptions we make about the unfolding over time of NATURE's possible strategies (Fig. 2).

If NATURE chooses the leftmost strategy then the pace of social change will be wrenching. The rightmost strategy would permit the pace of change to approximate the pace experienced in the first half of the 20th century. Although we might prefer that NATURE chooses the rightmost strategy (or fear the leftmost strategy), we cannot predict which strategy NATURE will confront us with because of the magnitude and global reach of the social changes rising oil prices will provoke. Increases in environmental degradation will show a similar pattern, perhaps with a different time scale.

For the purpose of illustration, we will concentrate on assumptions compatible with the continuation of the global market economy but allowing for the possibility of major social changes. Other sets of assumptions are possible. The point is that the decision maker will have to start with a set of assumptions because the future is unknowable. The assumptions we make to illustrate our decision methodology imply the emergence of new long term investment opportunities that are difficult to evaluate because of uncertainties and risk.

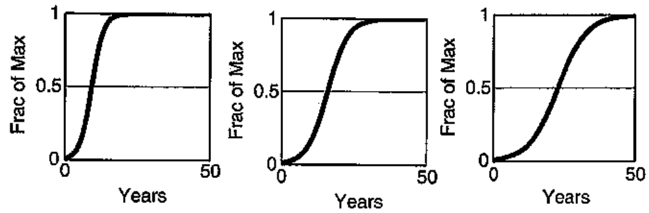

Fig. 2. The assumption is the *shape* of NATURE's strategy, not the specific path NATURE will follow. Also, for our purpose of illustration we have fixed the maximum price of oil at 100 times last year's price.

Attractive investments at a given oil price and level of environmental degradation will be aided or stressed by further price increases and deteriorations in the environment. These further changes in the underlying investment environment will force more social changes which will either help or hurt previous investment choices, i.e., the risk profiles of the investments will change.

3. Investment models

As we turn to the modeling task, making sense of this complicated decision environment, assumptions will help us simplify our picture of the world. No model can capture reality. However, good models based on thoughtful assumptions can capture a slice of reality large enough to inform our investment decisions.

Difficulties modeling uncertainty and risk have been with us since the pioneering work of Knight in the nineteen-twenties [1]. Our approach still maintains Knight's distinctions, i.e., uncertainty is unquantifiable variability (no assumption of the existence of a distribution) and risk is estimated in terms of quantifiable variability (variability with a distribution).

We consider investment return as a random function of the uncertainties. Uncertainties are left unquantified and modeled as number intervals. The uncertainties become our independent variables.

Second order statistics (quantification) of returns, which in turn are functions of the uncertainties, become the basis for our decision methodology aimed at achieving a balance between expected payoff and risk. We move from sets of numbers to sets of functions.

3.1. *Uncertainty and risk*

Two major approaches currently dominate the decision science literature dealing with uncertainty and risk [2]; however, both abandon Knight's distinctions. The first approach, associated with Bayesian statistics, quantifies uncertainty by introducing probability distributions. The second approach, associated with Fuzzy Set Analysis, introduces measures of belief. The measures are too weak to produce distributions as in Bayesian analysis but are strong enough to determine preferences. In our view both approaches are flawed by assuming additional information about the uncertainties, really subjective opinions elicited from the decision maker.

The price of light crude oil is modeled as an interval. The assumption of the sigmoid model for future oil prices allows us to think of the price of oil as a fraction of the maximum price, i.e., a surrogate for the price of oil is a number in [0,1].

The degradation of air quality standing for the degradation of the environment is modeled as an interval. The maximum level of air quality degradation is the level with a significant impact on human mortality. Again, a surrogate can be taken as the fraction of the maximum or a number in [0,1].

We are concerned with long term investments and so the price of oil and measure of the degradation of air quality are taken to be deterministic but unknowable, true uncertainties. Opposed to the Bayesians we do not assume the existence of a distribution for the uncertainties. Further, we do not wish to base decisions on measures of belief related to future values of the uncertainties as the fuzzy set analysts would.

Risk is defined in terms of the random variability of the criteria used to evaluate an investment alternative. We assume that for a fixed value of the uncertainties the criteria values have a distribution and risk is the probability of an undesirable outcome. Note that risk will be a function of the uncertainties.

3.2. *Investment returns*

The decision setting at this point is general enough to handle a number of investment problems depending on how we choose to handle the interplay of uncertainty and risk. For instance, we could consider structured portfolios optimized for one of several possible uncertain contingencies or scenarios.

Another class of decision problems are investment problems for platforms allowing for future adaptations. An example might be a power plant that is designed to be adaptable to multiple fuel sources: coal, natural gas, etc. Evaluating an investment in such a plant turns on the value of the flexibility designed into the plant at the additional capital cost.

Yet another class of problems are sequential decisions, i.e., decisions made at branchpoints depending on the outcomes of previous decisions. Here the difficulty is evaluating alternatives in terms of future options when we assume that the future is unknowable.

Continuing our illustrative example, we will consider the simplest investment problem. Annualized return rates for each investment alternative are random with distributions depending on the price of oil and the degradation of air quality modeled as uncertainties in the spirit of Knight. Recall that we are concerned with long term investments and, even though we assume that future prices and measures of degradation are increasing smooth functions of a special form representing NATURE's hidden strategy, we make no assumptions about the probability of a particular choice.

Each investment alternative will be evaluated in terms of estimated second order statistics, the mean and variance, of the return rates depending on the uncertainties. Note that the required statistics at this point in the model development do not depend on time but on the uncertainties, the price of oil and the measure of air quality degradation. Presumably estimating statistics is a simpler task than predicting the price of oil at some distant time in the future. Second order statistics represent the weakest assumptions we can make for investment performance. Since we are building models for investment performance in environments we have never experienced, the assumptions are speculative. However, keep in mind that there are no safe bets. We are looking for the best bet based on what we know and are willing to assume.

4. Decision models

We will present a game theory formulation of the decision problem. Game theory is not a way to obtain an otherwise unavailable problem solution but rather encourages strategic thinking aimed at resolving the decision problem. In particular, the game theory formulation introduces time into the decision process.

We will include a computational example with explicit computation of the mean and variance of the performance criteria using assumptions on a triangular distribution of return rates. The result of the game theory formulation is the computation of the mean and variance, μ and σ^2, for the "integrated" process which has a normal distribution. Finally, we introduce the decision variable $\mu - \alpha\sigma$, for some parameter α, which provides a balance of expected payoff and risk acceptable to the decision maker.

4.1. *A two person game: NATURE and PLAYER.*

We assume a single play. At the start of the game NATURE chooses as a strategy a vector function $\{(u(t),\ v(t)),\ 0 \le t < \infty\}$ (subject to some constraints) with values uncertainties from our basic decision model. Given the random performance X of one of PLAYER's available investment alternatives, let

$$Z_1(u, v) = \int_0^u \int_0^v EX(x, y)\mathrm{d}y\mathrm{d}x$$
$$Z_2(u, v) = \int_0^u \int_0^v X(x, y) - EX(x, y)(\mathrm{d}y)^{1/2}(\mathrm{d}x)^{1/2}$$

and

$$Z = Z_1 + Z_2.$$

We introduce a time dependent outcome for Player's choice with $\{Z(u(t), v(t)), 0 \le t < \infty\}$.

Given that future outcomes are discounted at a rate r, the score for the game is given by

$$V = \int_0^\infty \mathrm{e}^{-rt} Z(u(t), v(t))\mathrm{d}t.$$

The definitions and existence of the various integrals above follows from a straightforward application of the Central Limit Theorem [3]. A secondary result is the justification of our operator representations and simulation methods.

In order to provide PLAYER with a basis for making a rational choice from among available alternatives, we turn to the computation of the first two moments of V. We will discuss later the state of PLAYER's knowledge at the time of choosing an alternative.

4.1.1. *Computing the score and formal manipulations*

The computation of the first two moments of the score V involves simple integrals.

Keeping score. Assume that $\{(u(t), v(t)), 0 \le t < \infty\}$ is a strategy of NATURE.

$$\begin{aligned}\int_0^\infty \mathrm{e}^{-rt} Z_1(u(t), v(t))\mathrm{d}t &= \int_0^\infty \mathrm{e}^{-rt} \int_0^{u(t)} \int_0^{v(t)} EX(x, y)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\ &= \int_0^\infty \mathrm{e}^{-rt} \int_0^1 \int_0^1 (x \le u(t))(y \le v(t))EX(x, y)\mathrm{d}y\mathrm{d}x\mathrm{d}t \\ &= \int_0^1 \int_0^1 EX(x, y) \int_0^\infty \mathrm{e}^{-rt}(x \le u(t))(y \le v(t))\mathrm{d}t\mathrm{d}y\mathrm{d}x\end{aligned}$$

Let $Y = X - EX$.

$$\begin{aligned}\int_0^\infty e^{-rt} Z_2(u(t), v(t))dt &= \int_0^\infty e^{-rt} \int_0^{u(t)} \int_0^{v(t)} Y(x,y)(dy)^{1/2}(dx)^{1/2}dt \\ &= \int_0^\infty e^{-rt} \int_0^1 \int_0^1 (x \le u(t))(y \le v(t))Y(x,y)(dy)^{1/2}(dx)^{1/2}dt \\ &= \int_0^1 \int_0^1 Y(x,y) \int_0^\infty e^{-rt}(x \le u(t))(y \le u(t))dt(dy)^{1/2}(dx)^{1/2}.\end{aligned}$$

Thus

$$\begin{aligned}\int_0^\infty e^{-rt} Z(u(t), v(t))dt &= \int_0^1 \int_0^1 EX(x,y) \int_0^\infty e^{-rt}(x \le u(t))(y \le v(t))dtdydx \\ &\quad + \int_0^1 \int_0^1 (X(x,y) - EX(x,y)) \int_0^\infty e^{-rt}(x \le u(t)) \cdot (y \le v(t))dt(dy)^{1/2}(dx)^{1/2}.\end{aligned}$$

Easy consequences. For each $0 \le u(t), v(t) \le 1$, let $\ell(x,y) = \int_0^\infty e^{-rt}(x \le u(t))(y \le v(t))dt$. Then each of $\ell(\cdot, y)$ and $\ell(x, \cdot)$ is non-increasing on [0,1] and $\ell(0,0) = 1/r$. If

$$V = \int_0^\infty e^{-rt} Z(u(t), v(t))dt$$

then $\mu = EV = \int_0^\infty e^{-rt} EZ(u(t), v(t))dt = \int_0^1 \int_0^1 \bar{\mu}(x,y)\ell(x,y)dydx$ and

$$\begin{aligned}\sigma^2 &= E(V - EV)^2 \\ &= \int_0^1 \int_0^1 E(Z(x,y) - EZ(x,y))^2 \ell^2(x,y)dydx \\ &= \int_0^1 \bar{\sigma}^2(x,y)\ell^2(x,y)dydx\end{aligned}$$

where $\bar{\mu}(x,y) = EX(x,y)$ and $\bar{\sigma}^2(x,y) = E(X(x,y) - E(X(x,y)))^2$.

4.1.2. Computations based on NATURE's strategies

For fixed values x and y of the uncertainties, $X(x,y)$ should be a time rate. Note that

$$\ell(x,y) = \int_0^\infty e^{-rt}(x \le u(t))(y \le v(t))dt$$

is the total discounted time that $x \le u(t)$ and $y \le v(t)$ for $0 \le t < \infty$. Hence

$$\int_0^1 \int_0^1 EX(x,y)\ell(x,y)dydx$$

is the total time discounted contribution of X to the expected value of V.

$$\int_0^1 \int_0^1 (X(x,y) - EX(x,y))\ell(x,y)(dy)^{1/2}(dx)^{1/2}$$

is the total time discounted contribution of X to the random variability of V.

For our decision problem NATURE's strategy $(u(t), v(t))$, $0 \le t < \infty$ is drawn from order pairs of the family of sigmoid functions. We know "in general" how NATURE will act over time. We do not know the timing determined by a parameter c.

If $0 < u(0) < 1/2$, $0 < v(0) < 1/2$, $v(t) = 1/(1 + (1/v(0) - 1)e^{-ct})$, and $u(t) = 1/(1 + (1/u(0) - 1)e^{-ct})$ then let

$$\ell_c^1(x) = \begin{cases} 1/r & 0 \le x \le u_0 \\ \dfrac{1}{r}\left[\dfrac{u_0(1-x)}{x(1-u_0)}\right]^{r/c} & u_0 \le x \le 1 \end{cases}$$

and

$$\ell_c^2(y) = \begin{cases} 1/r & 0 \le y \le v_0 \\ \dfrac{1}{r}\left[\dfrac{v_0(1-y)}{y(1-v_0)}\right]^{r/c} & v_0 \le y \le 1. \end{cases}$$

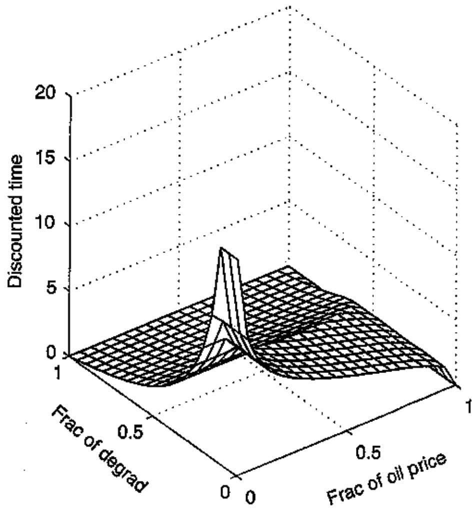

Fig. 3. An example of a surface $z = \ell_{(c_1,c_2)}(x, y)$.

Then

$$\ell_{(c_1,c_2)}(x, y) = \min(\ell^1_{c_1}(x), \ell^2_{c_2}(y))$$

$\ell_{(c_1,c_2)}(x, y)$ is the total discounted time that $x \le u_{c_1}(t)$ and $y \le v_{c_2}(t)$ for $0 \le t < \infty$. In Fig. 3 we arbitrarily set the discount rate at $r = 0.06$ and chose $c_1 = 0.2$ and $c_2 = 0.0667$.

Note that μ and σ^2 now both depend on the uncertain parameters c_1 and c_2, i.e.,

$$\mu(c_1, c_2) = \int_0^1 \int_0^1 \bar{\mu}(x, y)\ell_{(c_1,c_2)}(x, y)\mathrm{d}y\mathrm{d}x$$

and

$$\sigma^2(c_1, c_2) = \int_0^1 \int_0^1 \bar{\sigma}^2(x, y)\ell^2_{(c_1,c_2)}(x, y)\mathrm{d}y\mathrm{d}x.$$

By introducing the assumption in the form of NATURE's strategies (the one parameter family of sigmoids), we have reduced the uncertainties (the price of oil and the degradation of the environment) to uncertainties in the pace of change.

4.2. *A complete computational example*

To this point, our modeling efforts have concentrated on decision models. We must now take up the problem of constructing the model inputs, the description of our decision alternatives. We have to supply for each alternative the second order statistics $\bar{\mu}$ and $\bar{\sigma}^2$ as functions of the original uncertainties. This task is made easier by not having to contend with time as an independent variable. The second order statistics are independent of NATURE's strategy.

Consider the following description for a hypothetical investment alternative. Let X_1 denote the return rate and assume that X_1 has a triangular distribution with minimum $a = uv$, maximum $b = uv + 3$, and mode $c = 3 - 2uv$. Note that $\bar{\mu}_1 = EX_1 = \frac{a+b+c}{3} = 6/3 = 2$, i.e., $\bar{\mu}_1$ is constant, the expected rate of return for this possible investment is constant. Also, $\bar{\sigma}_1^2 = E(X_1 - EX_1)^2 = \frac{a^2+b^2+c^2-ab-ac-bc}{18} = \frac{13u^2v^2-9uv+9}{18}$.

The computation of μ_1 and σ_1^2. In order to simplify the computations and the example we will consider only contingencies where c_1 is chosen from {0.2,0.2857,0.5} and c_2 is chosen from {0.0667,0.0909,0.1429}. Thus we are restricting NATURE to choosing from among nine strategies. We stress that NATURE's choice is uncertain and the computation of μ_1 and σ_1^2 must be completed for each possibility (Tables 1 and 2).

The decision variable. For us, the risk associated with an investment is the probability of a bad outcome. Since the integrated process $Z = Z_1 + Z_2$ is normal, a bad outcome can be taken as $\{Z \le \mu - \alpha\sigma\}$ for some positive number α and the probability determined from a table lookup. The number α is determined by the decision maker's risk tolerance. Note that $\text{prob}\{Z \le \mu - \sigma\} \sim 0.163$. We should prefer an investment with rate of return X_1 to an investment with rate of return X_2 if $\mu_1 - \sigma_1 > \mu_2 - \sigma_2$ (Table 3).

Table 1
Computed values of μ_1 for nine contingencies.

c_1	c_2		
	0.0667	0.0909	0.1429
0.2	5.0027	6.3301	8.0469
0.2857	6.1832	7.9196	10.4836
0.5	7.4964	9.5918	13.0452

Table 2
Computed values of σ_1^2 for nine contingencies.

c_1	c_2		
	0.0667	0.0909	0.1429
0.2	5.1827	6.7728	9.5022
0.2857	7.9659	10.4092	15.0430
0.5	12.6081	16.1506	23.5280

Table 3
Computed values of $\mu_1 - \sigma_1$ for nine contingencies.

c_1	c_2		
	0.0667	0.0909	0.1429
0.2	2.7261	3.7276	4.9643
0.2857	3.3608	4.6932	6.6051
0.5	3.9456	5.5730	8.1946

Table 4
Computed values of $\mu_2 - \sigma_2$ for nine contingencies for a second possible investment choice.

c_1	c_2		
	0.0667	0.0909	0.1429
0.2	2.7405	3.7320	4.9582
0.2857	3.3744	4.6908	6.5823
0.5	3.9653	5.5714	8.1581

5. A decision methodology

Consider the following description for a second hypothetical investment alternative. Let X_2 denote the return rate and assume that X_2 has a triangular distribution with minimum $a = \frac{4-\sqrt{15}}{2}$, maximum $b = \frac{8+\sqrt{15}}{4}$, and mode $c = \frac{8+\sqrt{15}}{4}$. Note that $\bar{\mu}_2 = EX_2 = \frac{a+b+c}{3} = 6/3 = 2$, i.e., $\bar{\mu}_2$ is constant, the expected rate of return for this possible investment is constant. Also, $\bar{\sigma}_2^2 = E(X_2 - EX_2)^2 = \frac{a^2+b^2+c^2-ab-ac-bc}{18} = \frac{15}{32}$ is constant, the variance of the rate of return is also constant (Table 4).

Notice that $\mu_1(0.2, 0.0667) - \sigma_1(0.2, 0.0667) < \mu_2(0.2, 0.0667) - \sigma_2(0.2, 0.0667)$ and $\mu_2(0.5, 0.1429) - \sigma_2(0.5, 0.1429) < \mu_1(0.5, 0.1429) - \sigma_1(0.5, 0.1429)$. Thus neither $\mu_1 - \sigma_1$ nor $\mu_2 - \sigma_2$ dominates the other. Which investment should we prefer?

Extended multi-criteria decision analysis. In the end, models are assumptions about the real world and the decision maker will be guided by conclusions derived from these assumptions. Unfortunately, the conclusions about the future of an investment decision in this rapidly evolving new environment might not be testable in the present. Thus we must emphasize the rationality of the decision process.

There are several axiom systems for decision making (or preferences for decision alternatives) which guarantee a rational decision process. Too many "decision methodologies" which do not satisfy these simple axioms or pass a test for rationality purport to resolve complicated investment decisions. These methods are to be avoided.

Rational decisions can lead to unfavorable results because of the randomness of the outcomes. However, rational methods will produce consistent decisions and eliminate the erratic decisions which are doomed in the long run.

Savage offered seven postulates that a preference rule should satisfy in order that the preference rule should imply a subjective probability [4]. We are only concerned with the first four which guarantee that the decision methodology based on the preference rule is rational, i.e., produces consistent decisions. The second postulate "the Sure Thing Principle" is the crucial step. The other three are immediate.

Informally, if A is preferred to B and another decision criteria is added for which A and B score the same then we should still prefer A to B. The usual form of multi-criteria decision analysis passes Savage's criteria. However, adding uncertainties to the decision environment requires us to check whether Savage's first four postulates hold for the extended version of the methodology.

Briefly, in our setting evaluating a possible decision alternative returns a function of the uncertainties rather than a number requiring an extension of the usual multi-criteria decision analysis. In our example we are using $\mu - \sigma$, the decision variable, to compare alternatives.

We prefer an alternative with return rate X_1 to an alternative with return X_2 if $\mu_1 - \sigma_1 \geq \mu_2 - \sigma_2$ and if $\mu_1(c_1, c_2) - \sigma_1(c_1, c_2) > \mu_2(c_1, c_2) - \sigma_2(c_1, c_2)$ for some ordered pair (c_1, c_2) of uncertainty values, i.e., the first alternative performs at least as well as the second under all conditions and better for some particular set of conditions. If the two inequalities hold for $\mu_1 - \sigma_1$ and $\mu_2 - \sigma_2$ then we say that $\mu_1 - \sigma_1$ dominates $\mu_2 - \sigma_2$.

Following the pattern of multi-criteria decision analysis we identify the set $\mathcal{A}$ of efficient alternatives (the $\mu_i - \sigma_i$'s are non-dominated). If $\mathcal{A}$ contains only one member we are through, that member of $\mathcal{A}$ is the preferred choice. Otherwise, let $\mu - \sigma$ denote the upper envelope of the decision variables $\mu_i - \sigma_i$ associated with members of $\mathcal{A}$. Note that $\mu - \sigma$ will not correspond to any available alternative. The preferred choice is the alternative which minimizes $\sum_{(c_1,c_2)} |(\mu(c_1, c_2) - \sigma(c_1, c_2)) - (\mu_i(c_1, c_2) - \sigma_i(c_1, c_2))|^2$. This preference rule satisfies Savage's first four postulates. In the computational example, the first alternative is preferred.

6. Concluding remarks

Decision problems with more than two uncertainties might present themselves. We are presenting exploratory models which suggest that more than three uncertainties might make the analysis unwieldy. Rather than employing a large number of uncertainties, the modeler might attempt problem decomposition. We have found that considering a larger problem as a collection of smaller problems that result from a multilevel decomposition is a fruitful approach [5].

Producing the second order statistics for an investment alternative using triangular distributions seems to fit the usual thought pattern for many investors. Binomial distributions are also common. The game formulation for investment decisions can handle a range of possibilities.

Our interest is decision problems for a new era. Rapidly rising oil prices and degradation of the environment on the scale discussed reduces the relevance of past history. Still estimations of the second order statistics for an investment alternative based on history might have a place. We certainly experienced massive social changes which impacted investment valuations in the 20th century.

An important point in our presentation is the possibility for a wide ranging decision analysis. We can visualize alternate visions of the future. Indeed, environmental degradation could have a major impact on the price of oil resulting in precipitous price declines. We have modeled the two uncertainties as independent variables.

The decision models could be more complex in a different sense. We might have more than one criteria for evaluating an investment alternative and the criteria could "interact". The performance of an alternative measured by one criterion might influence the performance measured by another [5]. This additional complexity could be modeled within our framework and support still more decision analysis for the decision maker.

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